Individual Round

Name: ____________________

Team Name: ____________________

Instructions

1. Do not begin until instructed to by the proctor.
2. You will have 60 minutes to solve 10 problems.
3. Your score will be the number of correct answers. There is no penalty for guessing or incorrect answers.
4. No calculators or electronic devices are allowed.
5. All submitted work must be your own. You may not collaborate with anyone else during the individual round.
6. When time is called, please put your pencil down and hold your paper in the air. **Do not continue to write.** If you continue writing, your score may be disqualified.
7. Do not discuss the problems until all papers have been collected.
8. If you have a question or need to leave the room for any reason, please raise your hand quietly.
9. Good luck!

Acceptable Answers

1. All answers must be simplified as much as reasonably possible. For example, acceptable answers include $\sin(1°)$, $\sqrt{43}$, or π^2. Unacceptable answers include $\sin(30°)$, $\sqrt{64}$, or 3^2.

2. All answers must be exact. For example, π is acceptable, but 3.14 or 22/7 is not.

3. All rational, non-integer numbers must be expressed in reduced form $\pm\frac{p}{q}$, where p and q are relatively prime positive integers and $q \neq 0$. For example, $\frac{2}{3}$ is acceptable, but $\frac{4}{6}$ is not.

4. All radicals must be fully reduced. For example, $\sqrt{24}$ is not acceptable, and should be written as $2\sqrt{6}$. Additionally, rational expressions cannot contain radicals in the denominator. For example, $\frac{1}{\sqrt{2}}$ is not acceptable, and should be written as $\frac{\sqrt{2}}{2}$.

5. Answers should be expressed in base 10 unless otherwise specified.

6. Complex numbers should be expressed in the form $a + bi$, where both a and b are written in a form compliant with the rules above. In particular, no complex denominators are allowed. For example, $\frac{1+2i}{1-2i}$ should be written as $-\frac{3}{5} + \frac{4}{5}i$ or $\frac{-3+4i}{5}$.

7. If a problem asks for all solutions, you may give the answers in any order. However, no credit will be given if any solution is missing or any solution is given but not correct.

8. Angle measurements should be given in radians unless otherwise specified.

9. Answers must be written legibly to receive credit. Ambiguous answers may be marked incorrect, even if one of the possible interpretations is correct.

Individual Round

1. What is the last digit of the following expression?

$$2024^{2024} + 2023^{2023} + 2022^{2022} + 2021^{2021}$$

1. **8**

Solution: Since the last digit of the last digit of a times the last digit of b is equal to the last digit of $a \times b$. (We can check this quickly by using vertical calculation formula for multiplication). We also know that the last digit of $a+b$ is equal to the last digit of the sum of the last digit of a and the last digit of b. So the last digit of 2023^{2023} is the same as the last digit of 3^{2023}. By listing we can find a pattern that the last digit of 3^{4n} is 1, the last digit of 3^{4n+1} is 3, the last digit of 3^{4n+2} is 9, the last digit of 3^{4n+3} is 7. So the last digit of $3^{2023} = 3^{4\times 505+3}$ is 7. Similarly the last digit of 4^{2024} is 6, 2^{2022} is 4, and $1^{2020} = 1$. So the last digit of the expression is the last digit of $6+7+4+1 = 18$, which is $\boxed{8}$.

2. The first two terms of a sequence are $a_0 = -1$ and $a_1 = 1$, and it satisfies $a_n = (a_{n-1} + a_{n+1})/2$ for all $n \geq 1$. What is a_{347}?

2. **693**

Solution: Calculating the first few terms for the sequence gives $-1, 1, 3, 5, 7, \ldots$ with the general term $a_n = 2n - 1$ being easy to verify. Therefore $a_{347} = \boxed{693}$.

3. Anu has a playlist with 8 songs, $s_1, s_2, \ldots, s_8$. He wants to listen to s_3 after s_2, s_2 after s_1, and s_4 after s_1. How many arrangements of the playlist are there such that Anu can listen to the songs in the order he wants?

3. **5040**

Solution: Because Anu does not care when he listens to songs s_5 through s_8, we need only consider the relative order of songs s_1 through s_4. The possible orderings of these 4 songs force s_1 to be first. We are then free to place s_4 anywhere else in the relative ordering. Because s_3 must come after s_2, this forces the rest of the ordering, so there are 3 possible orderings compared to 24 if Anu did not place any restrictions on the orderings of the songs. Thus the total number of arrangements is equal to $8! \cdot \frac{3}{24} = 7! = \boxed{5040}$.

4. A regular octagon with side length a and a regular hexagon with side length b have the same area. Find the square of the ratio of their side lengths, a^2/b^2.

4. $\frac{3(\sqrt{6}-\sqrt{3})}{4}$

Solution: The area of the hexagon is $b^2 \cdot 3\sqrt{3}/2$ by multiplying the area of an equilateral triangle with side length b by 6. To find the area of the octagon we first find the area of the triangle formed by joining the center with two adjacent vertices, and multiply by 8. Call the side length of that triangle s. By the Law of Cosines,

$$a^2 = 2s^2(1 - \cos\frac{\pi}{4}) = 2s^2(1 - \sqrt{2}/2)$$

and by the sine area formula the area of the triangle is

$$\frac{1}{2}s^2 \sin\frac{\pi}{4} = s^2 \cdot \frac{\sqrt{2}}{4} = \frac{a^2}{2(1-\sqrt{2}/2)} \cdot \frac{\sqrt{2}}{4}.$$

Multiplying by 8 and equating with the expression with b gives

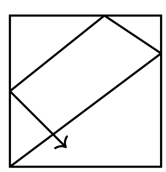
$$a^2 2(1+\sqrt{2}) = \frac{b^2 3\sqrt{3}}{2} \implies \frac{a^2}{b^2} = \boxed{\frac{3(\sqrt{6}-\sqrt{3})}{4}}.$$

5. Alex is participating in a tournament where each series is best-of-7, i.e. each game has a winner and the first person to win 4 games wins the series. After winning w series and losing ℓ series in the tournament, Alex has won 20 games and lost 12 games. Given this information, how many possible ordered pairs of (w, ℓ) are there?

5. **7**

Solution: Each of the w series wins has 4 games won and 0-3 games lost; similarly, each of the ℓ series lost has 4 games lost and 0-3 games won. Then $4w \le 20 \le 4w + 3\ell$ and $4\ell \le 12 \le 4\ell + 3w$ is necessary and sufficient. If $\ell = 0$, w must be 5. If $\ell = 1$, $4w \le 20 \le 4w + 3$, so w must still be 5. If $\ell = 2$, w can be 4 or 5. If $\ell = 3$, w can be $3, 4$, or 5. Clearly $\ell \le 12/4 = 3$, so there are $\boxed{7}$ possibilities in total.

6. A ball is launched on a flat, square table with slope of $\frac{11}{12}$ from the bottom left corner into the interior. When the ball reaches the edge of the table, it bounces back at an angle that is symmetric with the incidence angle. What is the total distance it travels before it reaches a corner for the first time?



6. $\sqrt{265}$

Solution: If the square is reflected across each edge that the ball collides with in its path, the path becomes a straight line in a grid of unit squares. The first time a line with slope 11/12 hits a lattice point is at $(12, 11)$ since they are coprime, so the distance traveled is $\sqrt{12^2 + 11^2} = \boxed{\sqrt{265}}$.

7. For each natural number greater than 1, we can perform the operation of dividing the number by 2 until it falls into the range $[1, 2)$. For example, performing this operation on 5 would yield $5 \to 2.5 \to 1.25$. Call a number *lucky* if the output of this operation on that number is greater than or equal to 1.5. How many natural numbers between 2 and 2024, inclusive, are lucky?

7. **1000**

Solution: Dividing a number by 2 is equivalent to bit shifting to the right by 1, so checking if a number is lucky amounts to checking if the two most significant bits are 11 (hence, the result of the operation on that number is at least $1.1_2 = 1.5$. For fixed k, the larger half of all binary k-digit numbers have a 1 in the second most significant bit, including the range $[1024 + 512, 2047]$. So the answer is $(2047 - 1)/2 - (2047 - 2024) = \boxed{1000}$.

8. A positive integer is said to be *square-containing* if the squares of each of its digits appear somewhere in the number. For example, if a number has the digit 5, it must contain the digit 2 immediately followed by the digit 5. For example, the number 1648 is not square-containing because it does not contain 36 even though it contains the squares of 1, 4, and 8. Without writing any leading zeros, what is the smallest square-containing number such that it contains at least one of each of the digits 0 through 9?

8. **25036497816**

Solution: Because our number must contain all of the digits and their squares, we know it contains the following sequences of digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, 1, 4, 9, 16, 25, 36, 49, 64, 81. Removing all duplicates and digits contained within other sequences, we get that our number must contain: 0, 7, 16, 25, 36, 49, 64, 81. To reduce the number of digits, we want to combine these to "double count" digits, e.g. 164 contains 16 and 64 so the 6 is double counted, as much as possible. There are two ways to get the maximum number of doubly-counted digits: 0, 7, 81649, 25, 36 and 0, 7, 25, 3649, 816. The smallest numbers you can create with each of these sets are 25036781649 and 25036497816. Therefore, the answer is $\boxed{25036497816}$.

9. Let n be an integer between 1 and 2024, inclusive. For positive integers m, let $f(m)$ denote the number of 1's in the binary representation of m. For example, for $26 = 2^4 + 2^3 + 2^1 = 11010_2$, we have $f(26) = 3$. For how many values of n does $f(f(f(n)))$ equal 1?

9. **1695**

Solution: First, instead of 2024, let's look at the next largest power of 2: $2048 = 2^{11}$. For all $n < 2^{11}$, we have $f(n) \leq 11$. For $f(n) \leq 11$, we have $f(f(n)) \leq 3$. Finally, for $f(f(n)) \leq 3$, we have $f(f(f(n))) = 1$ if $f(f(n)) = 1$ or 2, and $f(f(f(n))) = 2$ if $f(f(n)) = 3$. We will use complementary counting and count the number of n for which $f(f(f(n))) = 2$ then subtract that number from 2024. We have $f(f(f(n))) = 2 \implies f(f(n)) = 3 \implies f(n) = 7$ or 11. Notice that the number of $n < 2^{11}$ for which $f(n) = k$ is equal to $\binom{11}{k}$. This is because we are choosing k of the 11 binary digits to be equal to 1. Then, we have $\binom{11}{7} = 330$ values of n for which $f(n) = 7$ and $\binom{11}{11} = 1$ value of n for which $f(n) = 11$. However, we exclude 11111110000_2 and 11111111111_2 because they are larger than 2024. Therefore, our answer is $2024 - (330 + 1 - 2) = \boxed{1695}$.

10. You are next to a river with 2024 wolverines labeled from 1 to 2024, inclusive. They are all initially on the left side of the river. You spend the next 1012 days moving the wolverines back and forth across the river. On the first day, you take all the wolverines to the right side of the river. On the second day, you take every 3rd wolverine (the wolverines labeled $3, 6, 9, 12$, etc.) back to the left side of the river. In general, on the nth day, you take every wolverine with an index that is a multiple of $2n-1$ to the opposite side of the river. After the 1012 days have passed, how many of the wolverines are on the right side of the river?

10. **75**

Solution: First, notice that the wolverines that ended up on the right side of the river crossed the river an odd number of times. Note that the number of times the nth wolverine crossed the river is equal to the number of odd factors of n. Therefore, we need to count up how many n have an odd number of odd factors. Let $n = 2^{k_2} \cdot 3^{k_3} \cdot 5^{k_5} \cdot 7^{k_7} \cdot ... \cdot p^{k_p}$ be the prime factorization of n. Then, the number of odd factors of n equals $(k_3+1)(k_5+1)(k_7+1)...(k_p+1)$. (Note that (k_2+1) is excluded in order to ignore even factors of n). For n to have an odd number of odd factors, we need $(k_3+1)(k_5+1)(k_7+1)...(k_p+1)$ to be odd. This means that each of (k_3+1), (k_5+1), (k_7+1), ... , (k_p+1) needs to be odd, and therefore, each of k_3, k_5, k_7, ... , k_p needs to be even. This means that we need $3^{k_3} \cdot 5^{k_5} \cdot ... \cdot p^{k_p}$ to be a perfect square in order for all of the powers to be even. Furthermore, it is an odd perfect square because all of its prime factors are odd. Then, in order for n to have an odd number of odd factors, it has to be of the form $n = 2^{k_2} \cdot m^2$ for some odd m greater than or equal to 1. Instead of counting the number of possible n, we will count the number of possible m. We can proceed by casework. For $k_2 = 0$, we have $m^2 \leq 2024 \implies m \leq 43$. There are $\frac{1+43}{2} = 22$ odd numbers from 1 to 43. For $k_2 = 1$, we have $m^2 \leq \frac{2024}{2} = 1024 \implies m \leq 31$. There are $\frac{1+31}{2} = 16$ odd numbers from 1 to 31. We continue until $2^{k_2} > 2024$:

k_2	# of possible m
0	22
1	16
2	11
3	8
4	6
5	4
6	3
7	2
8	1
9	1
10	1

$22 + 16 + 11 + 8 + 6 + 4 + 3 + 2 + 1 + 1 + 1 = \boxed{75}$